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THE INTERACTION OF CRACKS IN BRITTLE FRACTURE. FORCE AND ENERGY APPROACHES†

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The quasistatic development of a system of cracks is investigated by writing the fracture criteria as variational inequalities, which enable the extension of the crack to be followed as a function of the time-like loading parameter. The interaction of cracks and their stability or avalanche-like growth are analysed. The similarity and differences between Irwin's and Griffith's criteria both in the usual and the refined formulations are discussed. © 2000 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a two-dimensional homogeneous body Ω weakened by a family of rectilinear cracks $M^j (j = 1, \ldots, J)$. The numbers $P^1, \ldots, P^N$ will be given to the crack tips positioned within Ω (for a boundary crack, only a single tip is marked, i.e. $N \leqslant 2J$). At each point P^n, its own system of polar coordinates (r_n, φ_n is introduced, and here $r_n = |x - P^n|$ and $\varphi_n = \pm\pi$ are the equations of the sides $M^j_\pm$ of a crack with a tip P^n. The stress intensity factors (SIFs) at P^n caused by a load p^0 on the external surface Γ of the body Ω will be denoted by K_1^{n0} and K_2^{n0} (for simplicity, we disregard the mass forces). We will introduce the time-like loading parameter $\tau \in [0, \tau_0)$ and represent the load in the form

$$p^\tau = p^0 + \tau p' + \tau^2 p'' + \ldots \tag{1.1}$$

The corresponding SIFs in the body $\Omega(0) = \Omega \backslash \{M^1 \cup \ldots \cup M^J\}$ with the initial damage configuration likewise become the sums

$$K_i^{n\tau} = K_i^{n0} + \tau K_i^{n'} + \tau^2 K_i^{n''} + \ldots, \quad n = 1, \ldots, N, \quad i = 1, 2 \tag{1.2}$$

If, during loading, at least one crack is a non-equilibrium crack, a family of cracks $\{M^1, \ldots, M^J\}$ will begin to develop. The aim of the present paper is to give, on the basis of various fracture criteria, formulations of problems that will enable the configuration of the family $\{M^j(\tau): j = 1, \ldots, J\}$ to be predicted at instant of time $\tau > 0$. Here it is assumed that cracks grow rectilinearly and cannot merge. The latter condition is ensured by the smallness of the parameter τ, and the former condition is discussed in Sections 3 and 7. The smallness of τ also enables us to replace Taylor series with quadratic approximations, i.e. to remove the "+ ..." from (1.1) and (1.2). The SIFs $K_i^n(\tau)$ corresponding to the load $p(\tau) = p^\tau$ and the displaced tip $P^n(\tau)$ can, of course, differ from $K_i^{n\tau}$. The extension of the crack M^j towards its tip P^n will be denoted by $h_n(\tau) = |P^n - P^n(\tau)|$. We will assume that all cracks are open (no contact of the sides) and cannot close (the fracture process is irreversible); in other words,

$$K_1^n(\tau) > 0, \quad h_n(\tau) \geqslant 0, \quad \tau \in [0, \tau_0), \quad n = 1, \ldots, N \tag{1.3}$$

The usual formulations $K_1^{n0} \geqslant K_{1c}$ and $\Delta U + 2\gamma(h_1 + \ldots + h_N) \leqslant 0$ of Irwin's and Griffith's criteria give the value of the critical load initiating the crack. Well-known approaches to the refinement of the formulations of the criteria (see [1, 2] for the force approach and [3, 4] for the energy approach) require reference to problems with unilateral constraints.

In Sections 3 and 4, the variational inequalities obtained by a *posteriori* analysis are reproduced (in other words, the conditions of equilibrium correlate with the intermediate and final positions of the cracks). Such inequalities enable us to describe the quasistatic development of the family $\{M^1, \ldots, M^J\}$,

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i.e. to find the functions $h_1, \ldots, h_N$ from (1.3) and to determine the conditions for the emergence of bifurcations or the transition to dynamic fracture. The basis for deriving variational inequalities is provided by various asymptotic formulae for the stress state of the body when the variation in crack lengths is arbitrary but small (Section 2). In both cases, data of the problems are expressed in terms of the same characteristics of the body at the instant of time $\tau = 0$, which makes it possible in Section 5 to compare the results obtained on the above assumptions on the basis of variational asymptotic models of various criteria. These models are not always equivalent, and here, under certain conditions, the model of Irwin's criterion predicts greater crack advance than the model of Griffith's criterion (Assumptions 2 and 3). Examples and generalizations are discussed in the final Sections 6–8.

2. ASYMPTOTIC FORMULAE

The displacement field in the body $\Omega(0)$ under the load (1.1) will be denoted by $u^\tau(x)$. Close to the apex P^n, the expansion

$$u^\tau(x) = \Lambda^\tau(x) + \sum_i \left\{K_i^{n\tau} r_n^{1/2} \Phi^{1i}(\varphi^n) + k_i^{n\tau} r_n^{3/2} \Phi^{3i}(\varphi^n)\right\} + O(r_n^2) \tag{2.1}$$

holds.

Throughout this paper, summation with respect to i and l is carried out from 1 to 2, and with respect to m and n from 1 to N. In (2.1), Λ^τ is a linear vector function of the variables x, Φ^{mi} are the known angular parts (see, for example, Section 1 in [5]), and $k_i^{n\tau}$ are the "lower" SIFs (they are not related to stress singularities but have an influence on crack advance). Furthermore, like relation (1.2), we have

$$u^\tau(x) = u^0(x) + \tau u'(x) + \tau^2 u''(x), \quad k_i^{n\tau} = k_i^{n0} + \tau k_i^{n'} + \tau^2 k_i^{n''} \tag{2.2}$$

We introduce the weighting functions ζ^{ni} with singularities at the point P^n (see [6, 7], and also Section 1 in [5]), i.e. the non-energy solutions of the homogeneous problem in the region $\Omega(0)$

$$\zeta^{ni}(x) = \delta_{nm} r_m^{-1/2} \Psi^{1i}(\varphi^m) + \sum_l A_{nm}^{il} r_m^{1/2} \Phi^{1l}(\varphi^m) + c^{nmi} + O(r_m) \tag{2.3}$$

where c^{nmi} are constant vectors, Ψ^{li} are the angular parts of the singular displacement fields (these are written out, for example, in Section 1 of [5]), A_{nm}^{il} are factors similar to SIFs that are geometrical characteristics of the damaged body, and δ_{nm} is the Kronecker delta. The weighting functions ζ^{ni} were introduced in [6, 7] to compute the SIFs

$$K_i^{n\tau} = \int p^\tau(x) \cdot \zeta^{ni}(x) ds_x \tag{2.4}$$

Here and below, integration is carried over the surface Γ of the body Ω.

Formula (2.4) is proved using integration by parts in the region $\Omega(0)$ with a cutout small circle $\{x: r_n < 0\}$ and then taking the limit as $\delta \to +0$. The same manipulations with ζ^{ni} and ζ^{ml} lead to the equations

$$A_{nm}^{il} = A_{mn}^{li} \tag{2.5}$$

We emphasize that all the quantities introduced relate to the initial position of the cracks and can therefore be used as data for the problem of the development of a family.

A slight modification of the procedure from [8] (see also Section 3 in [5] and Section 7.2 in [9]) enables us to construct the asymptotic forms of the solution $u(\tau; x)$ of the problem of the deformation of the body $\Omega(\tau) = \Omega \backslash \{M^1(\tau) \cup \ldots \cup M^J(\tau)\}$ by the load $p(\tau)$ (unlike $u^\tau(x)$, this is the real displacement field, taking account of crack growth). The "far" field, which serves as an approximation of $u(\tau; x)$ outside the neighbourhood of the crack tips, is expressed in terms of the outer expansion

$$u(\tau; x) \sim u^0(x) + \tau u'(x) + \alpha \sum h_n(\tau) \sum K_i^{n0} \zeta^{ni}(x) \tag{2.6}$$

In other words, the method of matched asymptotic expansions is used (see [10, 11]. etc.). Besides the correction $\tau u'(x)$ from (2.2), the right-hand side of (2.6) contains a linear combination of weighting functions which appears on account of crack elongation. The factor α equalizes the singular part $r^{-1/2}\Psi^{1i}(\varphi)$ with the kernel component $r^{1/2}\Phi^{1i}(\varphi)$ differentiated along the crack [12, 13] and in the isotropic case is computed from the Lamé constants λ and μ of the elastic material

$$\alpha = (\lambda + 2\mu)[2\mu(\lambda + \mu)]^{-1} \tag{2.7}$$

Since the explicit form of the angular parts $\Phi \ldots$ and $\Psi \ldots$ is not used in the present paper, according to well-known results [13], in some cases the examination of orthotropic bodies requires only a change in the factor (2.7).

The accuracy of approximation (2.6) is determined by the squares of the quantities τ and $h(\tau) = \max\{h_1(\tau), \ldots, h_N(\tau)\}$ (the maximum elongation). Due to singularities $O(r_n^{-1/2})$ close to the tip P^n, the outer expansion (2.6) is replaced by an inner expansion written using the fast variables $\xi^n = h_n(\tau)^{-1}[x - P^n(\tau)]$ and reflecting boundary layer effects. The explicity form of this expansion was presented earlier ([8] and Sections 3 and 5 in [5]). All of the asymptotic terms from (2.1) and (2.3) occur in this, but the smooth components $\Lambda^\tau(x)$ and c^{nmi} have no effect either on the far field (2.6) or on the following asymptotics form of the SIF

$$K_i^n(\tau) = K_i^{n0} + \tau K_i^{n'} + h_n(\tau)k_i^{n0}/2 + \alpha\sum_{m,l} A_{nm}^{il}h_m(\tau)K_l^{m0} + O(\tau^2 + h(\tau)^2) \tag{2.8}$$

The corrections of $h_n(\tau)k_i^{n0}/2$ and $\tau K_i^{n'}$ are governed by the displacement of the tip P^n and by the linear change in the external action $p^\tau \sim p^0 + \tau p'$, and the sum with respect to m and l describes the mutual influence of the growing cracks.

We will now examine the potential energy of deformation $U(\tau) = E(\tau) - R(\tau)$, where $E(\tau)$ is the elastic energy stored by the body $\Omega(\tau)$ and $R(\tau)$ is the work of external forces $p(\tau)$. Using relations (1.1), (2.6) and (2.4), we arrive at the well-known Griffith's formula, which is supplemented with the term $\tau U'$ produced by load variation

$$\begin{aligned} U(\tau) &= -\frac{1}{2}R(\tau) = -\frac{1}{2}\int p(\tau;x)\cdot u(\tau;x)ds_x = \\ &= U^0 + \tau U' - \frac{\alpha}{2}\sum_{n,i} h_n(\tau)K_i^{n0}\int p^0\cdot\zeta^{ni}ds_x + O(\tau^2 + h(\tau)^2) = \\ &= U^0 + \tau U' - \frac{\alpha}{2}\sum_{n,i} h_n(\tau)(K_i^{n0})^2 + O(\tau^2 + h(\tau)^2) \\ U^0 &= -\frac{1}{2}R^0 = -\frac{1}{2}\int p^0\cdot u^0 ds_x, \quad U' = -\frac{1}{2}\int (p^0\cdot u' + p'\cdot u^0)ds_x \end{aligned} \tag{2.9}$$

The refined formula (2.9) will be needed later, it can be derived and substantiated by constructing the successive terms of the inner and outer expansions for $u(\tau;x)$[5]. However, the same result can be achieved in a simpler way: relations (2.9) and (2.8) enable us to compute, with reliable accuracy, the derivative $\partial^2U/\partial\tau^2$. Thus

$$\begin{aligned} U(\tau) &= U^0 + \tau U' + \tau^2 U'' - \\ &- \frac{\alpha}{2}\sum_{n,i} h_n(\tau)K_i^{n0}\Big\{K_i^{n0} + 2\tau K_i^{n'} + \frac{1}{2}h_n(\tau)k_i^{n0} + \\ &+ \alpha\sum_{m,l} A_{il}^{nm}h_m(\tau)K_l^{m0}\Big\} + O(\tau^3 + h(\tau)^3) \\ U'' &= -\frac{1}{2}\int (p^0\cdot u'' + 2p'\cdot u' + p''\cdot u^0)ds_x \end{aligned} \tag{2.10}$$

We emphasize that the expression from the braces differs from the right-hand side of (2.8) by a factor of 2 for the quantity $\tau K_i^{n'}$.

3. A VARIATIONAL INEQUALITY BASED ON IRWIN'S CRITERION

Irwin's criterion implies that, close to the tips $P^1, \dots, P^N$, a mode-1 stress state (normal fracture) predominates, i.e. the SIFs $K_2^1, \dots, K_2^N$ vanish or are negligible, which is consistent with the assumption on the preservation of rectilinearity. Such a state can be realized for cracks lying on the same line, or for several similar ranks remote from each other.

We will introduce convenient notation. We will take the column $H = (H_1, \dots, H_N)^t$ as an unknown quantity, with the components [see (1.3)]

$$H_n(\tau) = h_n(\tau)K_1^{n0} \geqslant 0 \tag{3.1}$$

From the coefficients A_{nm}^{11} [see (2.3)] we form the NXN matrix A, it is symmetrical by virtue of (2.5). The scalar product in Euclidean space $\mathbf{R}^N$ will be denoted by $\langle,\rangle$. In accordance with formula (3.1), the column H belongs to the quadrant $(\mathbf{R}_+)^N$ (the plus subscript indicates non-negativity of the components).

According to the force criterion of fracture, the crack is at rest if K_1^n does not exceed the critical SIF value K_{1c}. This condition will be written in expanded form, tracking at each instant of time τ both stationary and moving tips

$$h_n(\tau) = 0 \Rightarrow K_1^n(\tau) \leqslant K_{1c} \tag{3.2}$$

$$h_n(\tau) > 0 \Rightarrow K_1^n(\tau) = K_{1c} \tag{3.3}$$

The final row can be explained as follows: the inequality $K_1^n(\tau) < K_{1c}$, denoting the crack arrest up to instant of time τ, is excluded from (3.3) because, owing to the smallness of τ, the quasistatic process of fracture is naturally considered to be continuous. Essentially the same implications were used previously [1, 2, 14].

Acting in accordance with the normal scheme (see, for example, [15]), we will transform formulae (3.2) and (3.3) into an inequality that holds for any $X_n \geqslant 0$

$$(K_1^n(\tau) - K_{1c})(H_n(\tau) - X_n) \geqslant 0 \tag{3.4}$$

This is actually a concise form of relations (3.2) and (3.3), since the latter follow from (3,4) if it is assumed that $X_n = H_n(\tau) \pm H_n(\tau)$ when $H_n(\tau) > 0$ and that $X_n > 0$ when $H_n(\tau) = 0$.

We replace $K_1^n(\tau)$ with the asymptotic representation from (2.8), sum inequalities (3.4) over $n = 1, \dots, N$, and, after rearranging the terms, we obtain

$$\alpha\langle AH(\tau), H(\tau) - X\rangle + \langle BH(\tau), H(\tau) - X\rangle \geqslant \langle F^I(\tau), H(\tau) - X\rangle, \quad \forall X \in (\mathbf{R}_+)^N \tag{3.5}$$

$$\begin{aligned} &B = \operatorname{diag}\{B_1, \dots, B_N\}, \quad F^I(\tau) = (F_1^I(\tau), \dots, F_N^I(\tau))^t \\ &B_n = (2K_1^{n0})^{-1}k_1^{n0}, \quad F_n^I(\tau) = K_{1c} - K_1^{n0} - \tau K_1^{n'} \end{aligned} \tag{3.6}$$

The algebraic variational inequality (3.5)(which we will term the inequality corresponding to Problem I) should be interpreted as a model of the force criterion of fracture that takes account of the load increment and interaction of the cracks. All the data of Problem I are determined from the linear part $p^0 + \tau p^1$ of the forces p^τ and the initial position of the cracks $M^1, \dots, M^J$. Using Eq. (3.1) for the solution $H(\tau)$, the crack extensions $h_1(\tau), \dots, h_N(\tau)$ are reproduced as a function of the loading parameter τ. The interaction of the cracks is described by matrix A. The characteristics of the initial stress state close to the crack tips occur in the diagonal $N \times N$ matrix B, and the right-hand side of F^I takes up the entire dependence on the loading process, i.e. on p'.

4. A VARIATIONAL INEQUALITY BASED ON GRIFFITH'S CRITERION

According to Griffith's criterion, a family of cracks develops such that, at any instant of time τ, the minimum total energy $U(\tau) + \Pi(\tau)$ is attained. Denoting the surface energy density by γ, we find the corresponding increment

$$\Delta\Pi(\tau) = 2\gamma(h_1(\tau) + \dots + h_N(\tau)) \tag{4.1}$$

We will adopt the same conditions for normal fracture as in Section 3. We will replace the increment of potential energy $\Delta U(\tau)$ in the minimization problem with its asymptotic representation from (2.10). Calculating the variation in the functional obtained for the arguments $H_1(\tau), \dots, H_N(\tau)$, we arrive at a variational inequality (which we will term the inequality corresponding to Problem G) that differs from (3.5) solely in the replacement of $F'(\tau)$ by

$$F^G(\tau) = (F_1^G(\tau), \dots, F_N^G(\tau))' \tag{4.2}$$

where

$$F_n^G(\tau) = \frac{2}{K_1^{n0}} \frac{\gamma}{\alpha} - \frac{1}{2} K_1^{n0} - \tau K_1^{n'} = \frac{1}{2}\left[\frac{K_{1c}^2}{K_1^{n0}} - K_1^{n0}\right] - \tau K_1^{n'} \tag{4.3}$$

$$\alpha K_{1c}^2 = 4\gamma \tag{4.4}$$

Equation (4.4) expresses the usual relationship between the strength characteristics K_{1c} and γ.

Note that the quantities p'' and u'' form (1.1) and (2.2) occurred in Eq. (2.10) but disappeared when the variation in the functional was computed, i.e. the model corresponding to Problem G acquired the same accuracy as model (3.5), and the loading (1.1) could be considered to be linear. Everything that was stated at the end of Section 3 applies equally to Problem G. Thus, the reduced formulation of the energy criterion of fracture enables us to find the distribution of the crack extensions as a function of the loading parameter τ and the number n of the tip. This fact, together with the results of [4, 14] on the variation in the front of the crack in three-dimensional bodies, disproves the view held by Si and Libowitz [16] concerning the inadequacy of Griffith's criterion and the impossibility of its strict formulation. The conclusions drawn in [16] were based on an analysis of the energy release rate which arises naturally but by no means replaces the entire criterion.

5. COMPARISON OF THE SOLUTIONS OF THE PROBLEMS

A standard treatment of the results of the investigation of the variational inequalities in Problems I and G leads to the following conclusions.

1. If the problem has only a trivial solution $H(\tau) = 0$, the system of cracks is stationary.
2. If a small (when $\tau \to +0$) solution exists and is unique, then the family of cracks develops quasistatically and is stable.
3. Bifurcations may be observed when there are several small solutions.
4. The absence of a solution or failure to comply with the requirement $H(\tau) = o(1)$ as $\tau \to +0$, even by one of the solutions, results in an avalanche-like growth of cracks and requires that dynamic effects be taken into account.

We will establish the sufficient condition for the existence or uniqueness of the solution of Problem I or Problem G. In view of the similarity of the notation of these two problems, we will refer below only to the variational inequality (3.5), and we will not indicate indices I and G for the columns $F^I(\tau)$ and $F^G(\tau)$.

Proposition 1. If the matrix $\alpha A + B$ is negative definite, a unique solution $H(\tau)$ of problem (3.5) exists, and the following limit holds

$$c|H(\tau)|^2 + \langle F(\tau)_+, H(\tau)\rangle \leq \langle F(\tau)_-, H(\tau)\rangle \leq |F(\tau)_-| \cdot |H(\tau)| \tag{5.1}$$

$$c > 0,\ F(\tau)_\pm = (F_1(\tau)_\pm, \dots, F_N(\tau)_\pm),\ F_n(\tau)_\pm = \frac{1}{2}(|F_n(\tau)| \pm F_n(\tau)) \geq 0$$

A proof of the solvability of a problem like (3.5) with the condition formulated can be found in any textbook on convex analysis (for example, in [15]). The necessary limit is obtained if it is assumed that $X = 0$ and noted that $F(\tau)_+ - F(\tau)_+$. In fact, we have

$$H(\tau)|^2 \leq \langle -(\alpha A + B)H(\tau),\ H(\tau)\rangle \leq -\langle F(\tau), H(\tau)\rangle = \langle F(\tau)_-, H(\tau)\rangle - \langle F(\tau)_+, H(\tau)\rangle$$

It must also be mentioned that the components of the columns $H(\tau)$ and $F(\tau)_\pm$ are non-negative.

Suppose the condition $\alpha A + B < 0$ is satisfied. If $F_n(\tau) \geqslant 0$, then $H(\geqslant) = 0$ and Situation 1 occurs. It is therefore reasonable to take as the starting point the instant of time $\tau = 0$, for which at least one of the SIFs K_1^{n0} is identical with K_{1c} [according to definitions (3.6) and (4.4), this critical criterion is the same for both criteria]. Then $F_n(\tau)_- \leqslant \tau(K_1^{n'})_+$ and by virtue of (5.1)

$$|H(\tau)| \leqslant c^{-1}\tau((K_1')_+ + \ldots + (K_1^{N'})_+)$$

Thus, the cracks grow steadily and quasistatically but remain stationary if all $K_1^{n'}$ are negative (a change in the load causes a reduction in the SIF).

We will now consider the effect due to the interaction of cracks; not only are non-equilibrium ($K_1^{n0} = K_{1c}$, $K_1^{n'} > 0$) cracks begin to grow but, as the parameter τ increases, those cracks for which the SIF is near-critical are also involved in fracture. In this case, in accordance with estimate (5.1), the equation $H_n(\tau) = 0$ (immobility of the tip P^n) can be guaranteed only if $F_n(\tau)_+ > |F(\tau)_-|$, i.e. when the SIF K_1^{n0} is much less than the critical value.

Since it is always the case that $K_1^{n0} \leqslant K_{1c}$, from formulae (3.6) and (4.4) we infer that

$$F_n^G(\tau) \geqslant F_n^I(\tau), \quad n = 1, \ldots, N$$

These inequalities enable us to compare the solutions $H^G(\tau)$ and $H^I(\tau)$ of Problem G and Problem I.

Proposition 2. Let $\alpha A + B < 0$ and let the off-diagonal elements of the matrix A be non-negative. Then the solutions of Problem I and Problem G are related by

$$H^I(\tau) \geqslant H^G(\tau) \tag{5.2}$$

Proof. We will assume that $X = H^I + H_+^0$, corresponding to Problem I in inequality (3.5), and that $X = H^G - H_+^0$ in the similar inequality for Problem G; here $H^0 = H^G + H^I$ (argument τ is not written). Combining these inequalities, we obtain

$$\alpha\langle AH^0, H_+^0\rangle + \langle BH^0, H_+^0\rangle \geqslant \langle F^G - F^I, H_+^0\rangle$$

Since B is a diagonal matrix, we have

$$\langle BH^0, H_+^0\rangle = \langle BH_+^0, H_+^0\rangle$$

Furthermore

$$\langle AH^0, H_+^0\rangle = \langle AH_+^0, H_+^0\rangle - \langle AH_-^0, H_+^0\rangle$$

the subtrahend being positive owing to the property of the off-diagonal elements in A. Thus

$$0 \geqslant -c|H_+^0|^2 \geqslant \langle(\alpha A + B)H_+^0, H_+^0\rangle - \langle AH_-^0, H_+^0\rangle \geqslant \langle F^G - F^I, H_+^0\rangle \geqslant 0$$

From this it follows that $H_+^0 = (H^G + H^I)_+ = 0$, which means that the required inequality (5.2) is established.

Under the conditions of Proposition 2, the model of Irwin's criterion (Problems I) predicts greater crack advance than the model of Griffith's criterion (Problem G), and here the following obvious limit holds

$$|H^I(\tau) - H^G(\tau)|^2 \leqslant C\max\{(K_{1c} - K_n^0)^2(K_n^0)^{-1}(H_n^I(\tau) - H_n^G(\tau))\}$$

Consequently, the discrepancy between $H^I(\tau)$ and $H^G(\tau)$ increases when predicting the start of cracks with considerable deviations of SIFs $K^0{}_n$ from the critical value. Both criteria lead to the same result if initially non- equilibrium cracks prove to be the only ones growing (primarily) this concerns isolated cracks, see Section 5 in [5]).

Proposition 3. We will assume that $\alpha A + B < 0$ and $H_n^I(\tau) = H_n^G\tau) = 0$ when $K_1^{n0} < K_{1c}$. Then $H^I(\tau) = H^G(\tau)$.

Proof. We will combine inequality (3.5) corresponding to Problem I, where $X = H^G$, and the similar inequality for Problem G, in which we will assume that $X = H^I$. We have

$$c\,|\,H^0\,|^2 < -\langle(\alpha A + B)H^0, H^0\rangle \leq \sum\,(F_n^I - F_n^G)(H_n^G - H_n^I)$$

All of the terms of the latter sum are zero since, by virtue of (3.5) and (4.2), $F_n^I = F_n^G$ when $K_I^{n0} = K_{Ic}$ and, according to the proposition, $H_n^G = H_n^I = 0$ when $K_I^{n0} < K_{Ic}$. Thus, $H^0 = H^G - H^I = 0$.

As indicated by Proposition 2, the equivalence of the variational-asymptotic models of Irwin's and Griffith's criteria established in Proposition 3 is not always satisfied, although Eq. (4.4) fixes the same equilibrium threshold of cracks. Crack growth under conditions of interaction can lead to divergence of the solutions $H^I(\tau)$ and $H^G(\tau)$ within the limits of accuracy of the models (when deriving variational inequalities from asymptotic formulae, the lower terms were removed). Since the bifurcations are very sensitive to perturbations of the data of the problem, in Situation 3 qualitatively differing patterns of crack development may be obtained.

The next assertion relates to the case of avalanche-like crack growth, i.e. to Situation 4.

Proposition 4. If the matrix $\alpha A + B$ is positive definite and all the components of the right-hand side of $F(\tau)$ are negative, variational inequality (3.5) has no solutions.

Proof. It is sufficient to note that, for both criteria, variational inequality (3.5) is equivalent to the problem of the stationary points of the functional

$$(\mathbf{R}_+)^N \ni H \mapsto -\langle(\alpha A + B), H\rangle + 2\langle F, H\rangle$$

It is obvious that, under the conditions indicated, there are no such points.

6. EXAMPLES

The effects discussed in the previous section can even be traced in the simplest problem of a single internal crack (i.e. $J = 1$ and $N = 2$ and P^1 and P^2 are the tips of the cut M^1). If the distance d from the centre O of the crack M^1 to the external contour Γ is much greater than its length $a = |P^1 - P^2|$, it is possible approximately to compute the 2×2 matrix A, replacing Ω with the entire plane $\mathbf{R}^2$ and using a representation by the weighting function in terms of the Kolosov–Muskhelishvili potential (see Section 11.3 in [17])

$$A = \frac{1}{4a\alpha}\left[\mathbf{A} + O\left(\frac{a^2}{d^2}\right)\right], \quad \mathbf{A} = \begin{Vmatrix} -1 & 2 \\ 2 & -1 \end{Vmatrix} \tag{6.1}$$

An estimate of the remainder in (6.1) is obtained using general results concerning problems in regions with a singularly perturbed boundary (see [11, 18] and also [5, 19, 20] for elasticity theory). Note that the matrix A is not sign definite – its eigenvalues are equal to 1 and –3. We assume that the region $\Omega(0) = \Omega\backslash M^1$ is symmetrical about the Cartesian axes Ox_1 and Ox_2. The symmetry of the load in relation to the axis of the crack Ox_1 ensures the absence of a shear mode in the stress state close to the tips P^1 and P^2. Furthermore, for the matrix A from (6.1), the relations

$$A_{11} = A_{22} < 0, \qquad A_{12} = A_{21} > 0 \tag{6.2}$$

are retained. Therefore, in subsequent calculations, we will- eliminate the small quantity $O([a/d]^2)$ from (6.1) – this will not qualitatively change the pattern of crack growth but only give small perturbations to the threshold values of the parameters.

Initially, let the load p^τ also be symmetrical about the Ox_2 axis so that $K_I^1(\tau) = K_I^2(\tau)$. Since, for lower SIFs k_I^{10}, integral representations similar to (2.4) hold (see [7] and also [5, 9]), by selecting suitable forces p^0 and p' it is possible to ensure any SIF values, in particular

$$K_I^{10} = K_I^{20} = K_{Ic}, \quad k_I^{10} = k_I^{20}, \quad K_I^{1\prime} = K_I^{2\prime} > 0 \tag{6.3}$$

Problem I or Problem G satisfies the conditions of Proposition 1 only if $2ak_I^{10} < -K_I^{10}$, when the crack develops symmetrically and quasistatically, and the unique solution acquires the form

$$H_1(\tau) = H_2(\tau) = 4\alpha\tau K_{\rm I}^{1\prime}(-K_{\rm I}^{10} - 2ak_{\rm I}^{10})^{-1} K_{\rm I}^{10} \tag{6.4}$$

In (6.4) the factor of $K_{\rm I}^{10}$ (positive) coincides with the extension of the crack at each of the tips. When $2ak_{\rm I}^{10} \geqslant -K_{\rm I}^{10}$, there are no solutions of (3.5) (dynamic growth). In fact, we note first that $H(\tau) = 0$ cannot be a solution. If there is a non-trivial solution, then inequality (3.5), in which $X_i = H_i(\tau) + 1$, leads to the contradiction

$$-(2aK_0^{10})^{-1}(K_{\rm I}^{10} + 2ak_{\rm I}^{10})(H_1(\tau) + H_2(\tau)) \geqslant \tau K_{\rm I}^{1\prime} > 0$$

In particular, $2ak_{\rm I}^{10} > -K_{\rm I}^{10}$ in the case of a homogeneous stress field, and it is for this reason that energy balance during the quasistatic displacement of cracks is impossible (apropos of an inhomogeneous field, see [21]). It is obvious that, when $K_{\rm I}^{1\prime} < 0$, the solutions of Problems I and G are trivial, i.e. the crack stops.

By Proposition 3. Irwin's and Griffith's criteria lead to identical results in the case of (6.3). We will now examine a loading p^τ such that the component p' possesses no symmetry about the axis Ox_2 and for small $s > 0$

$$K_{\rm I}^{10} = K_{\rm I}^{20}(1+s) = K_{1c}, \quad k_{\rm I}^{10} = k_{\rm I}^{20}(1+s), \quad K_{\rm I}^{1\prime} = K_{\rm I}^{2\prime} \tag{6.5}$$

For the condition $2ak_{\rm I}^{10} < -K_{\rm I}^{10}$ of the unique solvability of Problem I and Problem G, we have

$$\begin{aligned}
&F_1(\tau) = -\tau K_{\rm I}^{1\prime}, \quad F_2 = \delta - \tau K_{\rm I}^{1\prime} \quad (\delta = \delta_I \text{ or } \delta = \delta_G) \\
&\delta_I = \frac{s}{1+s}K_{1c}, \quad \delta_G = \left(1 + \frac{s}{2}\right)\frac{s}{1+s}K_{1c} \\
&H_1(\tau) = \tau c b^{-1}, \quad H_2(\tau) = 0 \text{ when } \tau \leqslant T \\
&H_1(\tau) = \frac{\tau c}{b-2} - \frac{8a\delta}{b^2-4}, \quad H_2(\tau) = \frac{\tau c}{b-2} - \frac{4a\delta b}{b^2-4} \text{ when } \tau \geqslant T \\
&b = 1 - 2ak_{\rm I}^{10}(K_{\rm I}^{10})^{-1} > 2, \quad c = 4aK_{\rm I}^{1\prime}, \quad T = \frac{\delta b}{b+2}(K_{\rm I}^{1\prime})^{-1}
\end{aligned} \tag{6.6}$$

Thus, in both criteria, only the tip P^1 (at which $K_{\rm I}^{10} = K_{1c}$) advances up to a certain instant of time, and the crack then grows in both directions. The rate of displacement of the tips are identical only because, according to assumption (6.5), $K_{\rm I}^{1\prime}/K_{\rm I}^{10} = K_{\rm I}^{2\prime}/K_{\rm I}^{20}$. In addition, the start of the apex P^2 within the framework of the model of Griffith's criterion is later than for the model of Irwin's criterion, since in (6.6)

$$T_I = \frac{s}{1+s}\frac{b}{2+b}K_{1c}(K_{\rm I}^{1\prime})^{-1}, \quad T_G = \left(1 + \frac{s}{2}\right)T_I$$

However, an error of the order of s^2 in determining T is beyond the scope of the accuracy of the model used. since, when comparing Problems I and G, the quantities $O(\tau^2)$ and $O[h(\tau)^2]$ were discarded. An increase in the parameters $s > 0$ and $b > 2$ delays the instant of connection of the tip P^2 to the fracture process. The functions indicated in (6.6) are continuous with respect to the parameter τ. Finally, in Irwin's criterion the tip P^2 begins to move until the hypothetical SIF $K_{\rm I}^{2\tau}$ from (1.2) reaches the critical value K_{1c}. The same is true for Griffith's criterion, since the number s in (6.5) is small.

In the previous examples, an alternative was observed: problem (3.5) is either unsolvable or has a unique solution. As can be seen from what follows, bifurcations can arise if the elements $A_{12} = A_{21}$ of the matrix from (6.1) are negative. Using the Papkovich–Neuber concept and the maximum principle, it is established that the off-diagonal elements of the matrix A are positive if the family of cracks is positioned on the line L but remote from the external surface $\partial\Omega$(the conditions of Proposition 2 are thereby ensured). The same holds true for a family containing one or two long boundary cracks on L, Thus, bifurcations are revealed perhaps only because of the interaction of cracks with the remaining part of the boundary of the body. Numerical calculations of SIFs in similar situations can be found in [22] and elsewhere.

Bifurcation effects were investigated in detail in [1, 23], but here we will confine ourselves to the simplest example: we will adopt conditions (6.3) and replace the matrix (6.1) by

$$A = \frac{1}{4a\alpha}\begin{Vmatrix} 1 & -2 \\ -2 & 1 \end{Vmatrix}$$

As earlier, it can be verified that when $2ak_{\mathrm{I}}^{10} \geqslant -K_{\mathrm{I}}^{10}$ there are no solutions for problem (3.5). When $2ak_{\mathrm{I}}^{10} < -3K_{\mathrm{I}}^{10}$, Proposition 1 guarantees the uniqueness of the solution

$$H_1(\tau) = H_2(\tau) = 4a\tau K_{\mathrm{I}}^{1\prime}(K_{\mathrm{I}}^{10} - 2ak_{\mathrm{I}}^{10})^{-1}K_{\mathrm{I}}^{10} \tag{6.7}$$

The same symmetrical solution (6.7) is retained when $2ak_{\mathrm{I}}^{10} < K_{\mathrm{I}}^{10}$, but when $2ak_{\mathrm{I}}^{10} \in [-3K_{\mathrm{I}}^{10}, -K_{\mathrm{I}}^{10})$ a further two solutions arise, which are asymmetrical

$$H_i(\tau) = 4a\tau K_{\mathrm{I}}^{1\prime}(-K_{\mathrm{I}}^{10} - 2ak_{\mathrm{I}}^{10})^{-1}K_{\mathrm{I}}^{10}, \quad H_{3-i}(\tau) = 0, \quad i = 1, 2$$

According to well-known results [1, 23], the symmetrical solution (6.7) is unstable.

7. GENERALIZATIONS – GRIFFITH'S CRITERION

We will drop the assumption that the SIF K_2^{n0} is small. Besides the column $H^{(1)}(\tau)$ with components (3.1), we will introduce the column $H^{(2)}(\tau) = [H_1^{(2)}(\tau), \ldots, H_N^{(2)}(\tau)]^t$ with the components

$$H_n^{(2)}(\tau) = h_n(\tau)K_2^{n0} \tag{7.1}$$

There is no need to introduce any constraints on the components (7.1) corresponding to the shear mode. Now the problem of minimizing the functional $\Delta U(\tau) + \Delta\Pi(\tau)$ reduces to a variational inequality of the form

$$\begin{gathered}\alpha\sum_{i,l}\langle A^{(i,l)}H^{(i)}(\tau),\ H^{(l)}(\tau) - X^{(l)}\rangle + \langle B^{(1)}H^{(1)}(\tau),\ H^{(1)}(\tau) - X^{(1)}\rangle \geqslant \\ \geqslant \langle F^{(1)}(\tau),\ H^{(1)}(\tau) - X^{(1)}\rangle + \langle F^{(2)}(\tau),\ H^{(2)}(\tau) - X^{(2)}\rangle \\ \forall X^{(1)} \in (\mathbf{R}_+)^N,\quad X^{(2)} \in \mathbf{R}^N\end{gathered} \tag{7.2}$$

where A^{ij} are $N \times N$ matrices with elements A_{nm}^{il} from (2.3). and the components of the N columns $F^{(1)}(\tau)$ and $F^{(2)}(\tau)$ and the elements of the diagonal matrix $B^{(1)}$ have the form

$$F_n^{(1)}(\tau) = (K_{\mathrm{I}}^{n0})^{-1}\{K_{\mathrm{I}c}^2 - (K_2^{n0})^2\} - K_{\mathrm{I}}^{n0} - \tau K_{\mathrm{I}}^{n\prime}, \quad F_n^{(2)}(\tau) = -\tau K_2^{n\prime}$$

$$B_n^{(1)} = (2K_{\mathrm{I}}^{n0})^{-1}\{k_{\mathrm{I}}^{n0} + k_2^{n0}K_2^{n0}(K_{\mathrm{I}}^{n0})^{-1}\}$$

For the problem in question it is possible to prove an assertion similar to Proposition 1. However, the interpretation of the solution $H^{(1)}(\tau), H^{(2)}(\tau)$ is not as obvious as in the case of Problem G. Of course, in accordance with (3.1), the extensions $h_1(\tau), \ldots, h_N(\tau)$ are re-established for the first column $H^{(1)}(\tau)$. In addition, the values of $h_n(\tau)$ found may contradict formula (7.1), since all of the quantities occurring in it have already been computed. In all likelihood, the discrepancy is due to the assumption of rectilinear crack growth – the presence of a shear component should cause distortion of the trajectories, but the question of the possibility of determining their shape from the solution $H^{(1)}(\tau), H^{(2)}(\tau)$ remains open (see the related papers [24–27]. However, a switch to examining curved cracks may alter the essential nature of the variational inequality by the addition of new unknown quantities.

8. GENERALIZATIONS – IRWIN'S CRITERION

The two-term asymptotic form (2.8) was used to construct variational inequality (3.5). The formulation of the fracture criterion can be refined on the basis of more complete asymptotic representations of the SIFs. For example, under the conditions of Section 3, a three-term asymptotic form is derived (see [4] and Sections 3 and 5 in [5])

$$K_{\rm I}^n(\tau) = K_{\rm I}^{n0} + \tau K_{\rm I}^{n\prime} + \tau^2 K_{\rm I}^{n\prime\prime} + \frac{1}{2} h_n(\tau) k_{\rm I}^{n0} + \alpha \sum_m A_{nm} H_m(\tau) + $$
$$+ \mathbf{K}_n(H;\tau) + O(\tau^3 + h(\tau)^3) \tag{8.1}$$

$$\mathbf{K}_n(H;\tau) = \alpha \sum_m A_{nm} h_m(\tau) \left\{ \tau K^{m\prime} + \alpha \sum_q A_{mq} H_q(\tau) + \frac{1}{4} h_m(\tau) k_{\rm I}^{m0} \right\} +$$
$$+ \frac{1}{2} h_n(\tau) \left(\tau k_{\rm I}^{n\prime} + \frac{3}{4} h_n(\tau) \varkappa_{\rm I}^{n0} \right) + \frac{\alpha}{2} \sum_m (h_n(\tau) D_{nm} + D_{mn} h_m(\tau)) H_m(\tau) \tag{8.2}$$

Here, the summation with respect to q is carried out from 1 to N, $\varkappa_1^{n0}$ is the coefficient of the singular component $r_n^{5/2}\Phi^{5.1}(\varphi^n)$ in an expansion of the form (2.1) of the solution u^0 (the formula for $\Phi^{5.1}$ is given in Section 1 of [5]). D_{nm} are the factors of $r_n^{3/2}\Phi^{3.1}(\varphi^m)$ in the revised expansions (2.3) of the weighting function ζ^{n1}, and in the representation of the non-linear operator (8.2) the replacement of $h_n(\tau)$ by $H_n(\tau)$ according to formula (3.1) is implicit. Following, the same procedure as in Section 3, using the asymptotic form (8.1) we can convert relations (3.2) and (3.3) into a problem with unilateral constraints

$$\langle(\alpha A + B)H(\tau),\ H(\tau) - X\rangle + \langle \mathbf{K}(H;\tau),\ H(\tau) - X\rangle \geqslant \langle F(\tau), H(\tau) - X\rangle,\ \ \forall x \in (\mathbf{R}_+)^N \tag{8.3}$$
$$F(\tau) = K_{\rm Ic} - K_{\rm I}^{n0} - \tau K_{\rm I}^{n\prime} - \tau^2 K_{\rm I}^{n\prime\prime}$$

containing the terms $\tau^2 K_1^{n''}$ and $\tau k_1^{n\prime}$ from (1.2) and (2.2), which were not called for earlier in (3.5).

The use of complicated asymptotic formulae should enable problem (8.3) to describe the development of a family of cracks over a greater range of values of the paramenter τ than problem I. In addition, the presence in (8.3) of the coefficients $\varkappa_1^{n0}$, D_{nm}, and the non-linear operator $\mathbf{K} = (\mathbf{K}_1, \ldots, \mathbf{K}_N)^t$ makes it difficult both to prepare the data and to solve the problem itself. Therefore, the step-by-step solution of variational inequality (3.5) with a re-calculation of the initial data is very likely to be a more economical approach that nonetheless ensures the same accuracy.

The revised formulation of Griffith's criterion, similar to (8.3), requires the construction of successive terms of asymptotic form (2.10) of the potential energy of deformation. They are extremely cumbersome and are not written out here. The variational inequalities that arise are investigated by the previous scheme.

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